Hofer-Zehnder capacity and length minimizing paths in the Hofer norm

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Abstract

We use the criteria of Lalonde and McDuff to determine a new class of examples of length minimizing paths in the group $Ham(M)$. For a compact symplectic manifold M of dimension two or four, we show that a path in $Ham(M)$, generated by an autonomous Hamiltonian and starting at the identity, which induces no non-constant closed trajectories of points in M, is length minimizing among all homotopic paths. The major step in the proof involves determining an upper bound for the Hofer-Zehnder capacity for symplectic manifolds of the type $(M \times D(a))$ where M is compact and has dimension two or four. In the appendix, we give an alternate proof of Polterovich's result that rotation in CP^2 and in the blow-up of CP^2 at one point is a length minimizing path with respect to the Hofer norm. Here we use the Gromov capacity and describe the necessary ball embeddings.

1 Background and Main Theorems

In this paper, we show that certain naturally occurring paths in the group $Ham(M)$ are length minimizing paths with respect to the Hofer norm. A length minimizing path ϕ_t for $0 \leq t \leq 1$ in $Ham(M)$ is a path which is an absolute minimum of the length functional among all paths from ϕ_0 to ϕ_1. The search for length minimizing paths is the logical extension of the work done on general geodesics, that is those which minimize length locally, by Bialy-Polterovich in [2], Ustilovsky in [18], and Lalonde-McDuff in [9].

Let (M, ω) be any symplectic manifold and $H_t : M \to \mathbf{R}$ for $0 \leq t \leq 1$ be a compactly supported time-dependent Hamiltonian function on M. The length $L(H)$ of H is defined to be

$$L(H) = \int_0^1 \max_{x \in M} H_t(x) - \min_{x \in M} H_t(x) dt.$$

The time-dependent Hamiltonian vector field X^H induced by H is the unique solution

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to the equation

$$i(X^H)\omega = -dH,$$

and its time t flow is denoted ϕ_t^H. The group $Ham^c(M)$ is the set of compactly supported time-one Hamiltonian maps on M:

$$Ham^c(M) = \{\phi_1^H \,\big|\, H_t : M \to \mathbf{R}\}.$$

When working on a compact manifold, we drop the superscript c and write only $Ham(M)$ since all diffeomorphisms have compact support.

Now, to any path ϕ_t for $0 \leq t \leq 1$ in $Ham^c(M)$, we can associate its generating time-dependent Hamiltonian H_t satisfying

$$\frac{d}{dt}\bigg|_{t=t_0}\phi_t = X_{t_0}^H(\phi_{t_0}^H).$$

The length of the path ϕ_t is defined as the length $L(H)$ of its generating Hamiltonian. The Hofer norm $\| \phi \|$ of $\phi \in Ham^c(M)$ is the infimum of the lengths of all of the paths from the identity to ϕ; hence, a globally length minimizing path from the identity to ϕ determines $\| \phi \|$. Although the Hofer norm is simply defined, it is difficult to calculate. One case in which it might be easiest to calculate $\| \phi \|$ is when there is a natural path from the identity to ϕ, e.g. a path induced by a circle action such as a rotation.

Lalonde and McDuff provide an important example of a globally length minimizing path when they show that rotation through π radians on S^2 is length minimizing in $Ham(S^2)$ [9]. This leads us to ask whether rotation of CP^2 through π radians is length minimizing in $Ham(\mathrm{CP}^2)$. In fact, by following the procedure outlined by Lalonde and McDuff in [9] using quasi-cylinders and capacities, we prove in the appendix that rotation on CP^2 and the blow-up of CP^2 at one point is indeed a length minimizing path. We work with Gromov capacity and show the necessary criteria are satisfied by constructing explicit embeddings of balls. These are independent proofs of results that Polterovich derives in [14] and [15].

However, the power of Gromov capacity to detect length minimizing paths is limited, and to obtain more general results we switch gears and work with the Lalonde-McDuff criteria paired with the Hofer-Zehnder capacity instead of the Gromov capacity. Before stating results, we need the follwing definition.

A path $\phi_t \in Ham(M)$ which starts from the identity has **no non-constant closed trajectory in time less than 1** if

$$\phi_{t_0}(x_0) = x_0 \text{ for some } t_0 \in (0, 1], x_0 \in M \Rightarrow \phi_t(x_0) = x_0 \ \forall t \in [0, 1].$$

The next theorem is the main result of this paper; note that the results about rotation in CP^2 and its blow-up from the appendix can also be derived as an application of this theorem.

Theorem 2.8 *Let (M, ω) be a compact symplectic manifold of dimension two or four. Let ϕ_t^H for $0 \leq t \leq 1$ be a path in $Ham(M)$ generated by an autonomous Hamiltonian $H : M \to \mathbf{R}$ such that ϕ_0^H is the identity diffeomorphism and ϕ_t^H has no non-constant closed trajectory in time less than 1. Then, the path ϕ_t^H for $0 \leq t \leq 1$ is length minimizing among all homotopic paths between the identity and ϕ_1^H.*

Theorem 2.8 generalizes Hofer's parallel result for $\mathbf{R}^{\mathbf{2n}}$. His proof that the flow of an autonomous Hamiltonian in $\mathbf{R}^{2n}$ which admits no non-constant closed trajectory in time less than 1 is a length minimizing path appears in Section 5.7 of [6]. In addition, Theorem 2.8 is an extension of Lalonde and McDuff's Theorem 5.4 from [9]. There they show that the conclusion holds if M has dimension two or if M is weakly exact. Siburg has generalized Hofer's result in another way; in [17] he extends the class of allowable Hamiltonians on $\mathbf{R}^{2n}$ to include time dependent functions as well as autonomous ones.

By the classification paper of Karshon, we know exactly what the semi-free Hamiltonian S^1 actions on a compact symplectic four manifold look like [7]. Hence, if M has dimension four and H actually generates a loop, i.e. the path ϕ_t^H represents a circle action, we know up to equivariant isomorphism the possible ways in which a ϕ_t^H that satisfies the hypotheses of Theorem 2.8 rotates M.

For the proof of Theorem 2.8, we follow the criteria for length minimizing paths from [9] using the Hofer-Zehnder capacity. Let $D(a)$ denote the open two-disk equipped with a symplectic form σ which satisfies $\int_D \sigma = a$. In order to complete the proof, we need to show that the Hofer-Zehnder capacity c_{HZ} satisfies the capacity-area inequality on all manifolds of the form $M \times D(a)$, equipped with the product symplectic form, where M is a symplectic manifold of dimension two or four. In [5], Hofer and Viterbo have proven that c_{HZ} satisfies this inequality for all $a > 0$ if the manifold M is weakly exact. This is a very restrictive condition which in particular excludes the case $M = \mathrm{CP}^2$ or the blow-up of CP^2. Hence, in Section 4, we return to the original proof of Hofer and Viterbo in [5] and modify it using the theory of J-holomorphic curves, proving:

Theorem 2.7 *Let (M, ω) be a compact symplectic manifold of dimension two or four. Then,*

$$c_{HZ}(M \times D(a), \omega \oplus \sigma) \leq a.$$

Remark Theorems 2.8 and 2.7 as they are now stated have limited scope. The restriction to manifolds of dimension two or four is required in order to deal with multiply covered curves on $M \times S^2$ at the end of Section 4. However, recent advances in the theory of $J-$holomorphic curves by Fukaya-Ono, Li-Tian, Liu-Tian, McDuff, Ruan, and Siebert, following ideas of Konstevitch, will most likely allow us to generalize to other dimensions. In particular, the methods that Liu and Tian use in [10] and McDuff's work in [11] that deal with stable virtual moduli spaces of curves can probably be used to extend Theorem 2.7 and Theorem 2.8 to include manifolds of all dimensions.

In related work, Polterovich examines a rotation, similar to the one considered in the appendix, on CP^2 and on the monotone manifold $(\widetilde{\mathrm{CP}^2}, \tau_{1/\sqrt{3}})$, the blow-up of CP^2 obtained by removing a ball of radius $\frac{1}{\sqrt{3}}$ centered at the point $[1:0:0]$. He examines the path ψ_t where

$$\psi_t[z_0 : z_1 : z_2] = [e^{2\pi i t} z_0 : z_1 : z_2].$$

He shows that the loop formed by ψ_t for $0 \leq t \leq 1$ is a length minimizing representative of its homotopy class in $Ham(\mathrm{CP}^2)$ in [14] and in $Ham(\widetilde{\mathrm{CP}^2})$ in [15]. Note that his results in [14] and [15] imply Theorems A.1 and A.5 of this paper; however his proofs rely on Gromov's K-area and a homomorphism combining the symplectic action and the

Maslov index. The proofs here using symplectic capacities and quasi-cylinders illustrate the criteria described in [9].

This paper is organized in the following way. The second section describes the criteria for length minimizing paths developed by Lalonde and McDuff in [9]. The third and fourth sections use $J-$holomorphic curve theory to prove Theorem 2.8 and Theorem 2.7. The appendix of this paper gives in full detail the ball embeddings which show that specific rotations on CP^2 and $\widetilde{\mathrm{CP}^2}$ are length minimizing .

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2 Criteria for length minimizing paths

We now briefly describe the theory that Lalonde and McDuff use to develop their criteria for length minimizing paths. In [9], they first derive a geometric way of detecting that $L(H_t) \leq L(K_t)$ for two Hamiltonians H_t and K_t on M . Then, they determine sufficient conditions involving symplectic capacities for this geometric requirement to be satisfied. We also describe the Gromov capacity and the Hofer-Zenhder capacity, two symplectic capacities used in this paper.

2.1 Results of Lalonde and McDuff

To begin, we must make a few definitions and set some notation. Suppose we have H, a compactly supported time dependent Hamiltonian function on the symplectic manifold (M^{2n}, ω). We may assume that for each t,

$$\min_{x \in M} H_t(x) = 0.$$

We write for the graph of H

$$\Gamma_H = \{(x, H_t(x), t)\} \subset M \times \mathbf{R} \times [0,1].$$

Now, let

$$h_\infty = \max_{x \in M, t \in [0,1]} H_t(x)$$

and suppose $\ell(t) : [0,1] \to [-\delta, 0]$ is a function which is negative and close to zero. A thickening of the region under Γ_H is

$$R_H^-(\frac{\nu}{2}) = \{(x,s,t) \Big| \ell(t) \leq s \leq H_t(x)\} \subset M \times [\ell(t), h_\infty] \times [0,1]$$

where $\int_0^1 -\ell(t)dt = \frac{\nu}{2}$. Similarly, we can define $R_H^+(\frac{\nu}{2})$ to be a slight thickening of the region above H:

$$R_H^+(\frac{\nu}{2}) = \{(x,s,t) \Big| H_t(x) \le s \le \mu_H(t)\} \subset M \times [0, \mu_H(t)] \times [0,1]$$

where $\mu_H(t)$ is a function dependent on H and t such that

$$\mu_H(t) \ge \max_{x \in M} H_t(x) \text{ and } \int_0^1 (\mu_H(t) - h_\infty)dt = \frac{\nu}{2}.$$

We define

$$R_H(\nu) = R_H^-(\frac{\nu}{2}) \cup R_H^+(\frac{\nu}{2}) \subset M \times \mathbf{R} \times [0,1].$$

For example, consider P defined on CP^2

$$P([z_0 : z_1 : z_2]) = \frac{\pi}{2} \frac{|z_0|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}.$$

Then,

$$\Gamma(P) = \left\{ \left([z_0 : z_1 : z_2], \frac{\pi}{2} \frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, t \right) \right\} \subset \mathrm{CP}^2 \times [0, \frac{\pi}{2}] \times [0,1]$$

$$R_P^-(\frac{\nu}{2}) = \left\{ ([z_0 : z_1 : z_2], s, t) \,\middle|\, \ell(t) \le s \le \frac{\pi}{2} \frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \right\}$$
$$\subset \mathrm{CP}^2 \times [\ell(t), \frac{\pi}{2}] \times [0,1]$$

$$R_P^+(\frac{\nu}{2}) = \left\{ ([z_0 : z_1 : z_2], s, t) \,\middle|\, \frac{\pi}{2} \frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \le s \le \mu_P(t) \right\}$$
$$\subset \mathrm{CP}^2 \times [0, \mu_P(t)] \times [0,1]$$

and

$$R_P(\nu) = \{([z_0 : z_1 : z_2], s, t) \,|\, \ell(t) \le s \le \mu_P(t)\} \subset \mathrm{CP}^2 \times \mathbf{R} \times [0,1].$$

Note that we can equip $R_H^-(\frac{\nu}{2})$, $R_H^+(\frac{\nu}{2})$, and $R_H(\nu)$ with the product symplectic form $\Omega = \omega \oplus ds \wedge dt$. We need the following definition from [9] which describes manifolds such as $(R_H(\nu), \Omega)$:

Definition 2.1 *Let (M, ω) be a symplectic manifold and D a set diffeomorphic to a disc in $(\mathbf{R^2}, ds \wedge dt)$. Then, the manifold $Q = (M \times D, \Omega)$ endowed with the symplectic form Ω is called a* **quasi-cylinder** *if*

(i) *Ω restricts to ω on each fibre $M \times \{pt\}$;*

(ii) *Ω is the product $\omega \oplus (ds \wedge dt)$ near the boundary $M \times \partial D$, and, in the case where M is non-compact, outside a set of the form $X \times D$ for some compact subset X in M.*

Note that for any Hamiltonian H, $(R_H(\nu), \Omega)$ is a quasi-cylinder symplectomorphic to $M \times D(L(H)+\nu)$ where $D(a)$ denotes the two-disk with area a. Since $\Omega = \omega \oplus ds \wedge dt$ everywhere, not just near the boundary, $R_H(\nu)$ is called a **split** quasi-cylinder. We define the **area** of any compact quasi-cylinder $(M \times D(a), \Omega)$ to be the number A such that

$$\text{vol } (M \times D(a), \Omega) = A \cdot \text{ vol } (M, \omega).$$

Note that if $(M \times D(a), \Omega)$ is split, its area is simply a. The area of $R_P(\nu)$, therefore, is $\nu + \frac{\pi}{2}$.

Now, suppose H_t and K_t are two Hamiltonians on M such that $\phi_1^H = \phi_1^K$ and the path ϕ_t^H for $0 \leq t \leq 1$ is homotopic (with fixed endpoints) to the path ϕ_t^K in $Ham^c(M)$. We may join Γ_K to Γ_H via the map

$$g(x,s,t) = (\phi_t^H \circ (\phi_t^K)^{-1}(x), s - K(x) + H(\phi_t^H \circ {\phi_t^K}^{-1}(x)), t).$$

This map g extends to a symplectomorphism of $R_K^+(\frac{\nu}{2})$, and we define

$$(R_{H,K}(\nu), \Omega) = R_H^-(\frac{\nu}{2}) \cup R_K^+(\frac{\nu}{2}).$$

Because the loop $\phi_t^H \circ {\phi_t^K}^{-1}$ is contractable in $Ham^c(M)$, Lalonde and McDuff are able to show that $(R_{H,K}(\nu), \Omega)$ is a quasi-cylinder diffeomorphic to

$$M \times \{s, t \in \mathrm{R}^2 \,\big|\, \lambda(t) \leq s \leq \mu_H(t)\} \cong M \times D(L(H)+\nu).$$

Note that $R_{H,K}(\nu)$ is not necessarily a split quasi-cylinder, and thus the area of $R_{H,K}(\nu)$ is not necessarily $L(H)+\nu$.

The key to the analysis in [9] is the following lemma, whose proof we include for the convenience of the reader.

Lemma 2.2 *(Lalonde-McDuff,[9], Part II, Lemma 2.1) Suppose that $L(K_t) < L(H_t) = A$. Then, for sufficiently small $\nu > 0$, at least one of the quasi-cylinders $(R_{H,K}(\nu), \Omega)$ and $(R_{K,H}(\nu), \Omega)$ has area $< A$.*

Proof: Choose $\nu > 0$ so that

$$L(K_t) + 2\nu < L(H_t),$$

and suppose first that M is compact. Evidently,

$$\begin{aligned} \text{vol}(R_{H,K}(\nu)) + \text{vol}(R_{K,H}(\nu)) &= \text{vol}(R_H(\nu)) + \text{vol}(R_K(\nu)) \\ &= (\text{vol}M) \cdot (L(H_t) + L(K_t) + 2\nu) \\ &< 2(\text{vol}M) \cdot L(H_t) \end{aligned}$$

where $R_H(\nu) = R_H^-(\frac{\nu}{2}) \cup R_H^+(\frac{\nu}{2})$. If M is non-compact, we may restrict to a large compact piece X of M and then take the volume. □

Lemma 2.2 tells us that if the area of both quasi-cylinders $(R_{H,K}(\nu), \Omega)$ and $R_{K,H}(\nu), \Omega)$ is greater than or equal to $L(H_t)$, then $L(H_t) \leq L(K_t)$. To develop their criteria for

length minimizing paths, Lalonde and McDuff use the theory of symplectic capacities to estimate the area of quasi-cylinders. A symplectic capacity is a function from the set of symplectic manifolds to $\mathbf{R} \cup \{\infty\}$ satisfying certain properties; in particular, it is a symplectic invariant. For more information on symplectic capacities, see [6]. Suppose we have chosen a particular capacity c and symplectic manifold (M, ω). We say the **capacity-area inequality** holds for c on M if

$$c(M \times D(a), \Omega) \leq \text{ area of } (M \times D(a), \Omega)$$

holds for all quasi-cylinders $(M \times D(a), \Omega)$. In the next section, we will give examples of manifolds and capacities that satisfy this condition. Although capacities are applied to symplectic manifolds, Lalonde and McDuff define the capacity of a Hamiltonian in the following way [9].

Definition 2.3 *The capacity* $\mathbf{c}(\mathbf{H})$ *of a Hamiltonian function* H_t *is defined as*

$$c(H) = \min\{\inf_{\nu>0} c(R_H^-(\frac{\nu}{2})), \inf_{\nu>0} c(R_H^+(\frac{\nu}{2}))\}.$$

Now, take a manifold M and a capacity c such that the capacity-area inequality holds for c on M, and suppose that we have a Hamiltonian $H_t : M \to \mathbf{R}$ for which

$$c(H) \geq L(H_t).$$

Then, for any Hamiltonian K_t generating a flow ϕ_t^K which is homotopic with fixed end points to ϕ_t^H (and thus has $\phi_1^K = \phi_1^H$), we can embed $R_H^-(\frac{\nu}{2})$ into $R_{H,K}(\nu)$ and $R_H^+(\frac{\nu}{2})$ into $R_{K,H}(\nu)$. Thus, we know

$$L(H_t) \leq c(H) \leq c(R_H^-(\frac{\nu}{2})) \leq c(R_{H,K}(\nu))$$

$$L(H_t) \leq c(H) \leq c(R_H^+(\frac{\nu}{2})) \leq c(R_{K,H}(\nu)),$$

with the last inequality in both lines holding by the monotonicity property of capacities. Since capacity-area inequality holds, we know that the areas of both quasi-cylinders $R_{H,K}(\nu)$ and $R_{K,H}(\nu)$ must be greater than or equal to their capacities and hence greater than or equal to $L(H_t)$. Therefore, by Lemma 2.2, $L(K_t) \geq L(H_t)$. This proves the proposition from [9]:

Proposition 2.4 *(Lalonde-McDuff, [9], Part II, Proposition 2.2) Let* M *be any symplectic manifold and* $H_{t\in[0,1]}$ *a Hamiltonian generating an isotopy* ϕ_t^H *from the identity to* ϕ_1^H*. Suppose there exists a capacity* c *such that the following two conditions hold:*

(i) $c(H) \geq L(H_t)$ *and*

(ii) *there exists a class* $\mathcal{S}$ *of Hamiltonian isotopies homotopic rel endpoints to* ϕ_t^H *,* $t \in [0, 1]$*, which is such that the capacity-area inequality holds (with respect to the given capacity* c*) for all quasi-cylinders* $R_{H,K}(\nu)$ *and* $R_{K,H}(\nu)$ *corresponding to Hamiltonians* $K_t \in \mathcal{S}$*.*

Then, the length of the path ϕ_t^H is minimal among all paths in $\mathcal{S}$.

Hence, to show that H_t generates a length minimizing path ϕ_t^H for $t \in [0,1]$ among all paths homotopic rel endpoints, we need only produce a capacity c that satisfies the above conditions (i) and (ii). In fact, Lalonde and McDuff show that if the capacity-area inequality holds for all split quasi-cylinders of the form $M \times D(a)$, then it also holds for all $R_{H,K}$ in Proposition 4.4 of [9]. Therefore, it will be enough to find a capacity that satisfies (i) and satisfies (ii) for all split quasi-cylinders, $M \times D(a)$. Our $\mathcal{S}$ will be the set of all Hamiltonians K_t where $\phi_1^K = \phi_1^H$ and ϕ_t^K is homotopic rel endpoints to ϕ_t^H.

2.2 Capacities

The symplectic capacities we will work with in this paper are the Gromov capacity, c_G, and the Hofer-Zehnder capacity, c_{HZ}. We recall their definitions for the convenience of the reader.

Definition 2.5 *Let (N,ω) be a symplectic manifold of dimension $2n$.*

(i) *The* **Gromov capacity**

$$c_G(N,\omega) = \sup\left\{\pi r^2 \,\middle|\, \begin{array}{c} \exists \text{ a symplectic embedding} \\ \phi : (B^{2n}(r),\omega_0) \to (N^{2n},\omega) \end{array}\right\}$$

where $(B^{2n}(r),\omega_0)$ is the open $2n$-dimensional ball with radius r endowed with the standard symplectic form.

(ii) *The* **Hofer-Zehnder capacity**

$$c_{HZ}(N,\omega) = \sup\{\max(H) \,|\, H \in \mathcal{H}_{ad}(N,\omega)\}$$

where $\mathcal{H}_{ad}(N,\omega)$ consists of all of the autonomous Hamiltonians on N satisfying the properties

(a) *There exists a compact set $\kappa \subset N \setminus \partial N$ depending on H so that $H \,|\, (N \setminus \kappa) = \max(H)$ is constant.*

(b) *There is a nonempty open set U depending on H such that $H \,|\, U = 0$.*

(c) $0 \le H(x) \le \max(H)$ *for all $x \in N$.*

(d) *All T-periodic solutions of the Hamiltonian system $\dot{x} = X_H(x)$ on N with $0 \le T \le 1$ are constant.*

To check that the capacity-area inequality holds on split quasi-cylinders for either of these capacities is a non-trivial procedure. By using J-holomorphic curve techniques, Lalonde and McDuff show in [9] that it holds for c_G on manifolds M, compact at ∞, which are of 4 dimensions or fewer or which are semi-monotone. Recently, they have shown that it holds for all M in [8] .

We know, then, that condition (ii) from Proposition 2.4 is satisfied for c_G on any manifold, and in particular on CP^2 endowed with the standard symplectic form τ_0 derived from the Fubini-Study metric . In the appendix, we use Propostion 2.4 and

c_G, construct specific embeddings of 6-balls, and show that rotation through π radians around the first coordinate in CP^2 and in $\widetilde{\mathrm{CP}^2}$ (the blow-up of CP^2 at the point $[1:0:0]$) is length minimizing among all homotopic paths. In addition we explain why c_G, for volume reasons, cannot be used to show the analagous rotation around the second coordinate in $\widetilde{\mathrm{CP}^2}$ is length minimizing.

Since c_{HZ} is not directly related to volume in the same way as c_G, the next natural step is to see if we can use c_{HZ} to show paths, in particular this rotation in the second coordinate of $\widetilde{\mathrm{CP}^2}$, are length minimizing. Thus we need to examine the conditions under which the capacity-area inequality (condition (ii) of Proposition 2.4) holds for c_{HZ}. Recall that a symplectic manifold (M, ω) is **weakly exact** if ω restricted to $\pi_2(M)$ is zero. The following theorem from [5] is quoted as Theorem 1.17 in [9]:

Theorem 2.6 *(Hofer-Viterbo) Let (M, ω) be a compact symplectic manifold which is weakly exact. Then for all $a > 0$,*

$$c_{HZ}(M \times D(a), \omega \oplus \sigma) \leq a.$$

However, $\widetilde{\mathrm{CP}^2}$ is not weakly exact, as the Hurewicz homomorphism is an isomorphism between $H_2(\widetilde{\mathrm{CP}^2}, \mathbf{Z})$ and $\pi_2(\widetilde{\mathrm{CP}^2})$. In order to eventually apply Proposition 2.4 to $\widetilde{\mathrm{CP}^2}$ using c_{HZ}, we will go back to the original proof of Theorem 2.6 and show that the restriction that M is weakly exact can be changed to M has dimension two or four. Hence, in the next section we arrive at

Theorem 2.7 *Let (M, ω) be a compact symplectic manifold of dimension two or four. Then for all $a > 0$,*

$$c_{HZ}(M \times D(a), \omega \oplus \sigma) \leq a.$$

Theorem 2.7 enables us to prove the following main result.

Theorem 2.8 *Let (M, ω) be a compact symplectic manifold of dimension two or four. Let ϕ_t^H for $0 \leq t \leq 1$ be a path in $Ham(M)$ generated by an autonomous Hamiltonian $H : M \to \mathbf{R}$ such that ϕ_0^H is the identity diffeomorphism and ϕ_t^H has no non-constant closed trajectory in time less than 1. Then, ϕ_t^H for $0 \leq t \leq 1$ is length minimizing among all homotopic paths between the identity and ϕ_1^H.*

Finally, a consequence of Theorem 2.8 and Proposition A.3 is that the path ϕ_t for $0 \leq t \leq 1$ in $Ham(\widetilde{\mathrm{CP}^2})$ given by

$$\phi_t[z_0 : z_1 : z_2] = [z_0 : e^{\pi i t} z_1 : z_2]$$

is length minimizing between the identity (ϕ_0) and rotation by π radians in the second coordinate (ϕ_1).

3 The capacity-area inequality for c_{HZ}

In the first part of this section, we analyze the proof of Theorem 2.6 which states sufficient conditions on M for c_{HZ} to satisfy the capacity-area inequality on M. Then,

in the second portion, we show that the weakly exact hypothesis in this theorem can be changed to dimension two or four.

3.1 Hofer and Viterbo's proof of Theorem 2.6

We now examine Hofer and Viterbo's proof of Theorem 2.6 to determine why they need the weakly exact condition [5]. Unfortunately, their notation is different from the notation in [9], so we will first need to provide some sort of dictionary to explain the theorem as they have stated it.

Let $[S^2, M]$ be the set of homotopy classes of maps from S^2 to M. We apply ω to such a class $\alpha \in [S^2, M]$ by evaluating ω on the representative of α in $H_2(M, \mathbf{Z})$. Define

$$m(M, \omega) = \inf\{\langle \omega, \alpha \rangle \,|\, \alpha \in [S^2, V],\ 0 < \ \langle \omega, \alpha \rangle\}.$$

Note that if M is weakly exact, $m(M, \omega) = \infty$. If for some particular class $\alpha \in H_2(M)$ we have $\langle \omega, \alpha \rangle = m(M, \omega)$, then α is called ω-minimal. The theorem of Hofer and Viterbo which is equivalent to Theorem 2.6 is

Theorem 3.1 *(Hofer-Viterbo, [5], Theorem 1.12) Let (M, ω) be a compact symplectic manifold and let σ be a volume form for S^2 such that $\int_{S^2} \sigma = a$ and*

$$0 < a \leq m(M, \omega).$$

Suppose $K : M \times S^2(a) \to \mathbf{R}$ is a smooth (time independent) Hamiltonian such that

$$K\mid_{\mathcal{U}(*)} = k_0 \text{ and } K\mid_{\mathcal{U}(M\times\{\infty\})} = k_\infty$$

for suitable neighborhoods of $M \times \{\infty\}$ and some point $* \notin M \times \{\infty\}$. Suppose

$$k_0 < k_\infty \quad \text{and} \quad k_0 \leq K \leq k_\infty.$$

Then, the Hamiltonian system $\dot{x} = X_K(x)$ on the symplectic manifold $(M \times S^2(a), \omega \oplus \sigma)$ possesses a non-constant T-periodic solution with

$$0 < (k_\infty - k_0)T < a.$$

The task now at hand is to see why Theorem 3.1 is equivalent to Theorem 2.6. Remember that

$$c_{HZ}(N, \omega) = \sup\{\max(H) \,|\, H \in \mathcal{H}_{ad}(N, \omega)\}$$

where $\mathcal{H}_{ad}(N, \omega)$ consists of all of the autonomous Hamiltonians on N satisfying the properties:

(a) There exists a compact set $\kappa \subset N \setminus \partial N$ depending on H so that $H \mid (N \setminus \kappa) = \max(H)$ is constant.

(b) There is a nonempty open set U depending on H such that $H \,|\, U = 0$.

(c) $0 \leq H(x) \leq \max(H)$ for all $x \in N$.

(d) All T-periodic solutions of the Hamiltonian system $\dot{x} = X_H(x)$ on N with $0 \leq T \leq 1$ are constant.

Clearly, proving Theorem 2.6 is the same as showing that any properly normalized Hamiltonian K on $M \times D(a)$ with $\max(K) > a$ has a non-constant orbit with period $T \leq 1$. In Theorem 3.1, Hofer and Viterbo consider the completion $M \times S^2(a)$ of $M \times D(a)$. For simplicity, we will also denote the symplectic form on $S^2(a)$ by σ. The neighborhood $U(M \times \infty) \subset M \times S^2(a)$ corresponds to a neighborhood of $\partial(M \times D(a))$ in Theorem 2.6. The hypotheses concerning the values k_0 and k_∞ in Theorem 3.1 correspond to the conditions (a) (b), and (c) describing the requirements for K to be a member of $\mathcal{H}_{ad}$. The hypothesis $0 < a \leq m(M, \omega)$ in Theorem 3.1 is satisfied for all a if and only if M is weakly exact. Finally, the quantity $k_\infty - k_0$ corresponds to $\max(K)$. Hence, to show the equivalence of the two theorems we need to suppose in Theorem 3.1 that $k_\infty - k_0 \geq a$ and show that we get a closed non-constant orbit of period $T \leq 1$. In fact, the conclusion of Theorem 3.1 tells us exactly that we get a non-constant orbit of period

$$T < \frac{a}{k_\infty - k_0},$$

so that if $k_\infty - k_0 \geq a$ then $T \leq 1$.

We eventually want to prove Theorem 3.1 without the hypothesis $a \leq m(M, \omega)$. Consider the symplectic manifold $(M \times S^2(a), \omega \oplus \sigma)$ and let $\mathcal{J}$ be the set of all smooth almost complex structures J compatible with $\omega \oplus \sigma$ on $M \times S^2(a)$. The original proof of Theorem 3.1 uses $J-$holomorphic curves with a split compatible almost complex structure $J \in \mathcal{J}$ on $M \times S^2(a)$ that is regular for the class $A = [\{pt\} \times S^2]$ in the sense of Theorem 3.1.2 of [13]. Hofer and Viterbo use a split J so that they can easily verify the condition $a < m(M, \omega)$ in certain settings. Since this condition is exactly the hypothesis we will remove, in this discussion we do not need to restrict ourselves to a split J. We will, however, need to impose more regularity conditions on J later.

After a J is fixed, the proof of Theorem 3.1 proceeds by determining the S^1-cobordism class of a certain moduli space of J-holomorphic spheres whose image is in $M \times S^2(a)$. This moduli space $\mathcal{M}(J)$ consists of the set of maps $u \in C^\infty(S^2, M \times S^2(a))$ that satisfy

$$[u] = [\{\text{pt}\} \times S^2(a)] = A \in H_2(M \times S^2(a), \mathbf{Z})$$

$$\int_D u^*(\omega \oplus \sigma) = \frac{a}{2} \text{ where } D = \{z \,\Big|\, |z| \leq 1\}$$

$$u(0) = \{*\}, \;\; u(\infty) \in M \times \{\infty\}$$

$$\overline{\partial}_J u = 0.$$

Hofer and Viterbo show the S^1-cobordism class of $\mathcal{M}(J)$ is not zero and hence a related family $\mathcal{C}$ of perturbed J-holomorphic spheres is not compact. Specifically,

$$\mathcal{C} = \{(\lambda, u) \in [0, \infty) \times \mathcal{B} \,|\, \bar{\partial}_J u + \lambda k(u) = 0\}$$

where $k(u)$ is basically a scaling of the gradient of K and $\mathcal{B}$ is the set of maps $u \in H^{2,2}(S^2, M \times S^2(a))$ that satisfy

$$[u] = A \in H_2(M \times S^2(a), \mathbf{Z})$$

$$\int_D u^*(\omega \oplus \sigma) = \frac{a}{2} \text{ where } D = \{z \Big| |z| \leq 1\}$$

$$u(0) = \{*\}, \ \ u(\infty) \in M \times \{\infty\}.$$

We can see that given a λ, the map u for $(\lambda, u) \in \mathcal{C}$ is almost fixed. Since J is regular, the dimension of the moduli space of perturbed $J-$holomorphic spheres of class A is $2c_1(A) + \dim(M) + 2 = 6 + \dim(M)$ ([13], Theorem 3.12). However, $\mathcal{C}$ does not consist of all of these spheres; the restrictions placed upon the elements in $\mathcal{B}$ reduce the dimension of $\mathcal{C}$ greatly. The first normalization condition on the area imposes a loss of 1 dimension. The next restriction, fixing the image of $\{0\}$, imposes a loss of $\dim(M) + 2$ dimensions. Finally, restriction the image of $\{\infty\}$ results in a loss of 2 dimensions. Hence, the set of spheres we are considering in the second factor of $\mathcal{C}$ will have dimension $6 + \dim(M) - 1 - (\dim(M) + 2) - 2 = 1$. This degree of freedom corresponds to rotation by S^1 of S^2. We are basically fixing the parametrization of u except for allowing this S^1 action. Note, then, that $\mathcal{C}$ is a two dimensional space: one dimension for the λ coordinate and one dimension which corresponds to this S^1 rotation.

Hofer and Viterbo analyze the noncompactness of $\mathcal{C}$ and show that it cannot be due to a bubbling off of perturbed J-holomorphic curves. Since there are no bubbles, there are uniform bounds on the derivatives of the u. They view the u not as maps from the sphere, but rather as maps from the non-compact cylinder $S^1 \times \mathbf{R}$. Hence, $\mathcal{C}$ consists of maps with finite energy whose domain is an infinitely long cylinder. In the same manner as in Floer theory, Hofer and Viterbo show the noncompactness of $\mathcal{C}$ produces a sequence of maps that converge to a closed non-constant orbit x which is a solution of the equation $\dot{x} = X_K(x)$.

When we remove the restriction $a \leq m(M, \omega)$, each of the steps in the proof of Theorem 3.1 goes through with only minor adjustments, except for the proof of the statement that there are no bubbles. It turns out, however, that this difficulty can be overcome. In the next section, we give a new proof that shows that it is still true generically that no sequence of elements in $\mathcal{C}$ converges to a bubble when we remove the area restriction if M has dimension two or four. We need the dimension restriction on M to rule out the possibility of multiply covered curves on $M \times S^2$.

3.2 Noncompactness in $\mathcal{C}$ cannot be due to bubbling

We will show that for generic $J \in \mathcal{J}$, the space of bubbles which are limits of sequences of elements in $\mathcal{C}$ is empty. We first show that for generic J, the space of cusp curves which have two components in empty.

There are five distinct types of two-component bubbles which are possible. We consider them separately. For each type, we will find a dense set of J so that the particular type does not occur; the intersection of these five sets will be our set of

regular J. The first case is when the point z_0 where the derivitave blows up in S^2 lies on the upper hemisphere but is not $\{\infty\}$; the other cases are when the point lies on the lower hemisphere, when the point lies on the equator, when the point is $\{0\}$, and when the point is $\{\infty\}$. We must separate the cases in this way to handle appropriately the restrictions of curves that lie in $\mathcal{C}$.

Let us now investigate the first case. We will represent the $\lambda k(u)$ perturbed component of the cusp curve by the class $A - Y$ and the J-holomorphic bubble by the class X. Let us for now assume that $X = Y$, and therefore that the homological sum of the two component classes is A. Note that this need not be the case: since we only consider simple cusp curves as limiting elements, we may have had to reduce a multiply covered curve and thus have lost some homology. We will discuss this later on and see that, since M has dimension two or four, it poses no obstacle.

Define the universal moduli spaces

$$\mu^\lambda(A - Y, \mathcal{J}) = \{(u, J) \,|\, u : S^2 \to M \times S^2(a), [\mathrm{Im}(u)] = A - Y,\ \bar{\partial}_J u + \lambda k(u) = 0\}$$

and

$$\mu(Y, \mathcal{J}) = \{(v, J) \,|\, v : S^2 \to M \times S^2(a), [\mathrm{Im}(v)] = Y,\ \bar{\partial}_J v = 0\}.$$

We will write $\mu^\lambda(A-Y, J)$ or $\mu(Y, J)$ when we wish to refer to the moduli space consisting of curves corresponding to a single J.

We must show that for a generic J, the subset of elements in $\mu^\lambda(A - Y, J) \times \mu(Y, J)$ which could be a limit of curves satisfying the restrictions of $\mathcal{C}$ and which are bubbles is empty. We are considering the first type of bubble where the point at which the derivative blows up to form the bubble lies in the upper hemisphere. A picture of the cusp curve is shown in Figure 1.

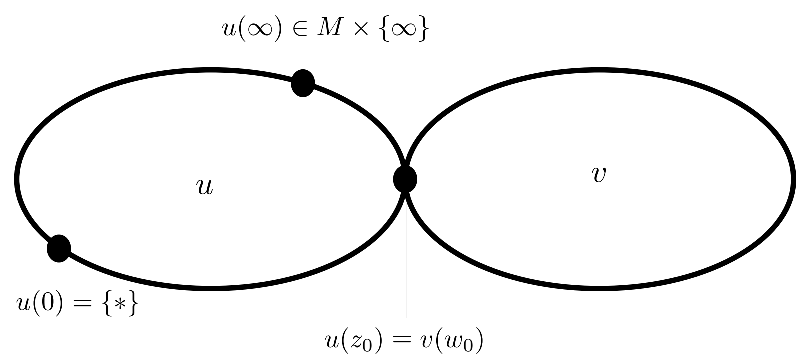

Figure 1: Bubbling in the upper hemisphere

Let $\mathcal{U}^\lambda$ be the space

$$\mathcal{U}^\lambda = \bigcup_{J \in \mathcal{J}} \mu^\lambda(A - Y, J) \times \text{ upper hemisphere of } S^2 \times \mu(Y, J) \times_G S^2.$$

Here $G = \mathrm{PSL}(2, \mathbf{C})$ is the six dimensional holomorphic reparametrization group of S^2. (Note that for different types of bubbles we will be able to quotient by different symmetry groups.) Define the space

$$\mathcal{U} = \{\lambda, \mathcal{U}^\lambda \,|\, \lambda \in [0, \lambda_\infty]\}$$

where λ_∞ is a constant described in [5] that depends on A and M. We let $\mathcal{U}_J$ be the restriction of $\mathcal{U}$ to a particular $J \in \mathcal{J}$. Next, we isolate the curves in $\mathcal{U}$ which are bubbles and can be limits of a sequence of elements in $\mathcal{C}$.

Consider the evaluation map ev where

$$ev : \mathcal{U} \to (M \times S^2)^4 \times \mathbf{R}$$

by

$$ev(\lambda, J, u, z_0, v, w_0) = \left(u(\infty), u(0), u(z_0), v(w_0), \int_D u^*(\omega \oplus \sigma) \right).$$

Let

$$\mathcal{D} = ev^{-1} \left((M \times \{\infty\}), \{*\}, \Delta, \frac{a}{2} \right)$$

where Δ stands for the diagonal in $(M \times S^2) \times (M \times S^2)$. We let $\mathcal{D}_J$ be the restriction of $\mathcal{D}$ to a particular $J \in \mathcal{J}$. Note that $\mathcal{D}_J$ consists exactly of the elements that are bubbles and could be the limit of a sequence of elements in $\mathcal{C}$. Our aim is to prove that, for a generic J, $\mathcal{D}_J$ is empty. Recall that although the curves in $\mathcal{C}$ do not have a full reparametization group acting on them, they do have an S^1 action. Hence, if $\mathcal{D}_J$ is non-empty, it must be of dimension at least one to account for this symmetry. We will show that generically $\mathcal{D}_J$ has dimension zero, and therefore it must be empty.

Of course for the analysis to make sense in this infinite dimensional setting, $\mathcal{U}$ needs to be a Banach manifold. Hence we must restrict the set of almost complex structures $\mathcal{J}$ to contain only those with sufficient smoothness and require that the curves u and v belong to an appropriate Sobolev space. These specific notions are described explicitly in Propositions 6.2.2 and 3.4.1 of [13].

Proposition 3.2 *There exists a set of complex structures $\mathcal{J}_0 \subset \mathcal{J}$ of second category such that for $J \in \mathcal{J}_0$, $\mathcal{D}_J$ is empty.*

Proof: We follow the steps in the proof of Theorem 6.3.2 from [13], using three lemmas. The first shows that ev is transversal onto the set

$$\left((M \times \{\infty\}), \{*\}, \Delta, \frac{a}{2} \right).$$

Then, the second, proves that the projection map π from $\mathcal{U}$ onto $\mathcal{J}$ is a Fredholm operator. Hence, there will be a set $\mathcal{J}_{reg}$ of regular values of π such that for $J \in \mathcal{J}_{reg}$, $\mathcal{U}_J$ is a manifold. We prove that the point $(\lambda, J, u, z_0, v, w_0)$ is a regular point of π exactly when ev restricted to $\mathcal{U}_J$ is transversal to $\left((M \times \{\infty\}), \{*\}, \Delta, \frac{a}{2} \right)$ at the point $ev(\lambda, J, u, z_0, v, w_0)$. Therefore, for $J \in \mathcal{J}_{reg}$, the submanifold $\mathcal{D}_J \subset \mathcal{U}_J$ has its expected codimension. The third lemma states that for J in some subset $\mathcal{J}_0 \subset \mathcal{J}_{reg}$ this codimension is equal to the dimension of $\mathcal{U}_J$ and hence $\mathcal{D}_J$ is empty.

Lemma 3.3 *The map ev is transversal to*

$$\Big((M \times \{\infty\}), \{*\}, \Delta, \frac{a}{2}\Big) \subseteq (M \times S^2)^4 \times \mathbf{R}.$$

Proof: Define the map

$$ev^{z_0,w_0} : \bigcup_{\lambda \in [0,\lambda_\infty]} \mathcal{J} \times \mu^\lambda(A - Y, J) \times \mu(Y, J) \to (M \times S^2)^4 \times \mathbf{R}$$

by

$$ev^{z_0,w_0}(\lambda, J, u, v) = (u(\{\infty\}), u(0), u(z_0), v(w_0), \int_D u^*(\omega \oplus \sigma)).$$

It suffices to show that for some pair

$$(z_0, w_0) \in (\text{upper hemisphere of } S^2 - \{\infty\} \times S^2),$$

the map ev^{z_0,w_0} is transversal onto

$$\Big((M \times \{\infty\}), \{*\}, \Delta, \frac{a}{2}\Big) \subseteq (M \times S^2)^4 \times \mathbf{R}.$$

Let $\pi_i : (M \times S^2)^4 \times \mathbf{R} \to (M \times S^2)$ be projection onto the ith $M \times S^2$ factor and let $\rho : (M \times S^2)^4 \times \mathbf{R} \to \mathbf{R}$ denote projection onto the last factor. Since transversality is a local condition and the points $\{0\}, \{\infty\}$, and z_0 are separated, it is enough to show that

$$e_1 = \pi_1 \circ ev^{z_0,w_0} \text{ is transversal to } M \times \{\infty\} \subset M \times S^2$$
$$e_2 = \pi_2 \circ ev^{z_0,w_0} \text{ is transversal to } \{*\} \in M \times S^2$$
$$e_{34} = \pi_3 \times \pi_4 \circ ev^{z_0,w_0} \text{ is transversal to } \Delta \subset (M \times S^2)^2$$

and

$$e_5 = \rho \circ ev^{z_0,w_0} \text{ is transversal to } \frac{a}{2}.$$

We now recall a theorem from [13] (Theorem 6.1.1). Let B be any class in $H_2(M \times S^2)$. For $x_0 \in S^2$, they define the map

$$e^{x_0} : \mu(B, \mathcal{J}) \to M \times S^2 \text{ by } e^{x_0}(u, J) = u(x_0).$$

Theorem 3.4 *(McDuff-Salamon) For any point x_0, the map e^{x_0} is a submersion onto $M \times S^2$.*

This theorem is stated for unperturbed curves, but its proof applies to the perturbed case as well. Hence, it directly implies that e_1 and e_2 are submersions and therefore certainly transversal. To show that e_{34} is transversal to Δ, note that the normal bundle to Δ at the point $(q, q) \in (M \times S^2)^2$ is spanned by $0 \oplus T_q(M \times S^2)$. By applying Theorem 3.4 we see $e_4 = \pi_4 \circ ev^{z_0,w_0}$ is a submersion. Therefore, e_{34} is indeed transversal to Δ. Finally, e_5 is transversal to $\frac{a}{2}$ simply because the area over D of the pull back of the symplectic form by the curve u has neither a local maximum nor a local minimum at $\frac{a}{2}$. □

Lemma 3.5 *There exists a set of second category $\mathcal{J}_{reg} \subset \mathcal{J}$, so that the codimension of $\mathcal{D}_J$ in $\mathcal{U}_J$ is $4n+7$ for all $J \in \mathcal{J}_{reg}$.*

Proof: Consider the projection map $\pi : \mathcal{U} \to \mathcal{J}$. Note that $(\pi)_*$ is onto, so its cokernel is 0 and hence finite dimensional. At the point $(\lambda, J, u, z_0, v, w_0) \in \mathcal{U}$, the kernel of $(\pi)_*$ consists of vectors

$$\{(\hat{\lambda}, Z, \xi_u, \hat{z}, \xi_v, \hat{w}) \,\big|\, Z = 0\}.$$

The ξ_u and ξ_v directions contribute finitely many dimensions no matter which J is chosen, and the other directions contribute a total of five dimensions. Hence the kernel is finite dimensional and π is Fredholm.

Let $\mathcal{J}_{reg} \subset \mathcal{J}$ denote the set of almost complex structures which are regular values of π. It is a set of second category. For $J \in \mathcal{J}_{reg}$, we know $\mathcal{U}_J$ is a manifold of the expected dimension.

Now, we show that the point $(\lambda, J, u, z_0, v, w_0)$ is a regular point of π if and only if at this point the restricted evaluation map

$$ev : \mathcal{U}_J \to (M \times S^2)^4 \times \mathbf{R}$$

is transversal to the set $\left((M \times \{\infty\}), \{*\}, \Delta, \frac{a}{2}\right)$.

We already know from Lemma 3.3 that the set of vectors

$$ev_*(\lambda, J, u, z_0, v, w_0)(\hat{\lambda}, Z, \xi_u, \hat{z}, \xi_v, \hat{w})$$

is transversal to $\left((M \times \{\infty\}), \{*\}, \Delta, \frac{a}{2}\right)$, and now we must explain why the subset of these vectors with $Z = 0$ (corresponding to keeping J constant) is still transversal. By Lemma 3.3 and the linearity of ev_* we see

$$\begin{aligned} T_{ev(\lambda,J,u,z_0,v,w_0)}((M \times S^2)^4 \times \mathbf{R}) &= \mathrm{Span}\left(\mathrm{Im}(ev_*) + T_{ev(\lambda,J,u,z_0,v,w_0)}\left((M \times \{\infty\}), \{*\}, \Delta, \tfrac{a}{2}\right)\right) \\ &= \mathrm{Span}(\mathrm{Im}(ev_*\big|_{Z=0}) + \mathrm{Im}(ev_*\big|_S) + \\ &\quad T_{ev(\lambda,J,u,z_0,v,w_0)}\left((M \times \{\infty\}), \{*\}, \Delta, \tfrac{a}{2}\right)). \end{aligned}$$

Here S is the set of tangent vectors that satisfy $\hat{\lambda} = \xi_u = \hat{z} = \xi_v = \hat{w} = 0$; that is the set of vectors for which all of the components except possibly the one in the Z direction are zero. However, we claim that $(\lambda, J, u, z_0, v, w_0)$ is a regular point of π if and only if

$$\mathrm{Span}\left(\mathrm{Im}(ev_*\Big|_S) + T_{ev(\lambda,J,u,z_0,v,w_0)}\left((M \times \{\infty\}), \{*\}, \Delta, \frac{a}{2}\right)\right) =$$

$$\mathrm{Span}\left(T_{ev(\lambda,J,u,z_0,v,w_0)}\left((M \times \{\infty\}), \{*\}, \Delta, \frac{a}{2}\right)\right).$$

Therefore, at a regular point of π,

$$T_{ev(\lambda,J,u,z_0,v,w_0)}((M \times S^2)^4 \times \mathbf{R}) = \mathrm{Span}\left(\mathrm{Im}(ev_*\big|_{Z=0}) + T_{ev(\lambda,J,u,z_0,v,w_0)}\left((M \times \{\infty\}), \{*\}, \Delta, \tfrac{a}{2}\right)\right)$$

and ev restricted to $\mathcal{U}_J$ is transversal. To prove the claim, note that for $(\lambda, J, u, z_0, v, w_0)$ to be a regular point of π means that for any $Z_0 \in T_J\mathcal{J}$, there exists a tangent vector

$$(\hat{\lambda}, Z_0, \xi_u, \hat{z}, \xi_v \hat{w}) \in T_{(\lambda,J,u,z_0,v,w_0)}\mathcal{U}.$$

In other words, for any $Z_0 \in T_J\mathcal{J}$,

$$ev_*(\lambda, J, u, z_0, v, w_0)(\hat{\lambda}, Z_0, \xi_u, \hat{z}, \xi_v \hat{w}) \in T_{ev(\lambda,J,u,z_0,v,w_0)}\left((M \times \{\infty\}), \{*\}, \Delta, \frac{a}{2}\right).$$

Thus, adding the vectors $\mathrm{Im}(ev_*|_S)$ to the set

$$T_{ev(\lambda,J,u,z_0,v,w_0)}\left((M \times \{\infty\}), \{*\}, \Delta, \frac{a}{2}\right)$$

do not change its span, because these vectors only have Z components and all Z components are already accounted for in the set.

Hence, for $J \in \mathcal{J}_{reg}$, ev restricted to $\mathcal{U}_J$ is transversal to $((M \times \{\infty\}), \{*\}, \Delta, \frac{a}{2})$. The inverse image of this set under ev, called $\mathcal{D}_J$, will be a manifold of the same codimension.

$$\begin{aligned}\text{codimension of } \mathcal{D}_J &= \text{codimension of } ((M \times \{\infty\}), \{*\}, \Delta, \tfrac{a}{2}) \\ &= 2 + (2n+2) + (2n+2) + 1 \\ &= 4n + 7.\end{aligned}$$

□

Finally, we calculate the dimension of $\mathcal{U}_J$.

Lemma 3.6 *There exists a set of second category $\mathcal{J}'_{reg} \subset \mathcal{J}$ so that for $J \in \mathcal{J}'_{reg}$, the dimension of $\mathcal{U}_J$ is $4n + 7$.*

Proof: We recall a theorem from [13] [Theorem 3.1.2]. Let B be any class in $H_2(M \times S^2)$.

Theorem 3.7 *(McDuff - Salamon) There exists a set of second category $\mathcal{J}'_{reg}(B) \subset \mathcal{J}$, such that for $J \in \mathcal{J}'_{reg}(B)$ the moduli space $\mu(B, J)$ is a smooth manifold of dimension $2c_1(B) + 2n + 2$.*

If we let our classes be $A - Y$ and Y, we see that for

$$J \in \mathcal{J}'_{reg}(A - Y) \cap \mathcal{J}'_{reg}(Y) = \mathcal{J}'_{reg},$$

we have

$$\begin{aligned}\dim \mathcal{U}_J &= 1 + (2c_1(A - Y) + 2n + 2) + 2 + (2c_1(Y) + 2n + 2) + 2 - 6 \\ &= 2c_1(A) + 4n + 3 \\ &= 4n + 7.\end{aligned}$$

□

Let $\mathcal{J}_0 = \mathcal{J}_{reg} \cap \mathcal{J}'_{reg}$. For $J \in \mathcal{J}_0$, $\mathcal{U}_J$ is a manifold of dimension $4n + 7$ in which $\mathcal{D}_J$ has codimension $4n + 7$. Hence, for these J, $\mathcal{D}_J$ will have dimension zero and in fact be empty as described earlier. Note that $\mathcal{J}_0$ is of second category since it is the intersection of two second category sets . Thus, we have proven Proposition 3.2. □

Proposition 3.8 *Suppose (M, ω) is a compact symplectic manifold of dimension two or four. Then, there exists a set of second category of regular almost complex structures on $M \times S^2$ for which the space of bubbles which are limits of sequences of elements in $\mathcal{C}$ will be empty.*

Proof: Proposition 3.2 tells us that for generic J, the space of such bubbles that are cusp curves with two components, neither of which is multiply covered, where the bubble is of a given homology class Y and is formed by the derivative blowing up at a point on the upper hemisphere, is empty. To deal with other types of bubbling in a two component cusp curve is similar. We must be careful in defining the evaluation map to use the correct domain, quotienting out by the appropriate reparametrization group, and set the area condition of the last component properly. Here are the precise variations, indexed by the point on the sphere at which the derivative blows up:

(i) Lower Hemisphere Change the domain of ev by setting

$$\mathcal{U}^\lambda = \bigcup_{J \in \mathcal{J}} \mu^\lambda(A - Y, J) \times \text{ lower hemisphere of } S^2 \times \mu(Y, J) \times_G S^2.$$

Use

$$\int_D u^*(\omega \oplus \sigma) + \int_{S^2} v^*(\omega \oplus \sigma)$$

for the final component in the map ev.

(ii) Equator Change the domain of ev by setting

$$\mathcal{U}^\lambda = \bigcup_{J \in \mathcal{J}} \mu^\lambda(A - Y, J) \times \text{ equator of } S^2 \times \mu(Y, J) \times_G S^2.$$

Use

$$\int_D u^*(\omega \oplus \sigma) + \int_D v^*(\omega \oplus \sigma)$$

for the final component in the map ev.

(iii) The point $\{0\}$ Change the domain of ev by setting

$$\mathcal{U}^\lambda = \bigcup_{J \in \mathcal{J}} \mu^\lambda(A - Y, J) \times \mu(Y, J) \times_{G_0} S^2$$

where G_0 is the four dimensional set of holomorphic maps from S^2 to itself which fix $\{0\}$, and let

$$ev(\lambda, J, u, v, w_0) = \left(u(\infty), v(0), u(0), v(w_0), \int_D u^*(\omega \oplus \sigma) + \int_{S^2} v^*(\omega \oplus \sigma) \right).$$

(iv) The point $\{\infty\}$ Change the domain of ev by setting

$$\mathcal{U}^\lambda = \bigcup_{J \in \mathcal{J}} \mu^\lambda(A - Y, J) \times \mu(Y, J) \times_{G_\infty} S^2$$

where G_∞ is the four dimensional set of holomorphic maps from S^2 to itself which fix $\{\infty\}$, and let

$$ev(\lambda, J, u, v, w_0) = \left(v(\infty), u(0), u(\infty), v(w_0), \int_D u^*(\omega \oplus \sigma)\right).$$

The proofs of the transversality and dimension results for these cases are analogous to the case examined in Proposition 3.2. Hence, for each case there is a set of second category of regular J for which there will be no two component bubbles of a certain homology class Y which are not multiply covered. Now, for each case, take the J which are in the set for all Y, i.e. the intersection over the countable set of possible homology classes Y. The interesection of these five sets (one for each case) is the set of regular almost complex structures described in the proposition.

To show that multiple bubbles would not occur, the argument from the proof of Proposition 3.2 can be modified. For each additional bubble, we would increase the number of homology classes used to form $\mathcal{U}_J$ by 1 and increase the number of S^2 used by 2. (See Theorem 6.3.2 from [13]). This adds $2n + 2 + 4 = 2n + 6$ to the dimension of $\mathcal{U}_J$, and we may reduce by the six dimensional reparametrization group G to get $2n$ added dimensions. The transversality results would carry through. The codimension of $\mathcal{D}_J$ with one added bubble would increase by $2n + 2$. Hence, the codimension of $\mathcal{D}_J$ would be greater than the dimension of $\mathcal{U}_J$, so $\mathcal{D}_J$ will be empty.

Finally, we must deal with the possibility of multiply covered curves. Without loss of generality, assume that the cusp curve has two components: the $\lambda k(u)$ perturbed $J-$holomorphic component of class $A - Y$ and the $J-$holomorphic bubble component of class X. Suppose that X has been reduced from the multiply covered dX where $dX - Y = 0$ in homology for some positive integer $d > 1$. Since M has dimension two or four, $M \times S^2$ has dimension less than or equal to six. Therefore, all classes representable by a $J-$holomorphic or perturbed $J-$holomorphic curve give a nonnegative integer when paired with the first Chern class. In particular, dX is representable so

$$c_1(X) = \frac{1}{d} \cdot c_1(dX) \geq 0.$$

This gives us

$$c_1(A) = c_1(A - Y) + d \cdot c_1(X) > c_1(A - Y) + c_1(X) = c_1(A - Y + X).$$

Consider the case fully explained in Proposition 3.2; the others are identical. When we imitate the proof of Lemma 3.6, we see that the space we would consider as the domain of the evaluation map is $\mathcal{U}'_J = \{\lambda, \mathcal{U}_J^{\lambda'}\}$ where

$$\mathcal{U}_J^{\lambda'} = \mu^\lambda(A - Y, J) \times \text{ upper hemisphere of } S^2 \times \mu(X, J) \times_G S^2.$$

We calculate

$$\begin{aligned}
\dim \mathcal{U}'_J &= 1 + (2c_1(A - Y) + 2n + 2) + 2 + (2c_1(X) + 2n + 2) + 2 - 6 \\
&= 3 + 2c_1(A - Y + X) + 4n \\
&< 3 + 2c_1(A) + 4n \\
&= 7 + 4n.
\end{aligned}$$

Note, however, that the codimension of $\mathcal{D}_J$ will still be $7+4n$. Hence, again, generically $\mathcal{D}_J$ will be empty. We can deal with the case when $A-Y$ has been reduced from a multiply covered component in a similar manner. $\Box$

4 Proofs of main theorems

We will restate the theorems here as we prove them.

Theorem 2.7 *Let (M,ω) be a compact symplectic manifold of dimension two or four. Then,*

$$c_{HZ}(M \times D(a), \omega \oplus \sigma) \leq a.$$

Proof: Fix an almost complex structure $J \in \mathcal{J}$ on $M \times S^2$ so that Proposition 3.8 holds. Note that Proposition 3.8 implies that Theorem 3.1 and hence Theorem 2.6 hold for M, if M has dimension two or four. This completes the proof. $\Box$

Theorem 2.8 *Let (M,ω) be a compact symplectic manifold of dimension two or four. Let ϕ_t^H for $0 \leq t \leq 1$ be a path in $Ham(M)$ generated by an autonomous Hamiltonian $H : M \to \mathbf{R}$ such that ϕ_0^H is the identity diffeomorphism and ϕ_t^H has no non-constant closed trajectory in time less than 1. Then, ϕ_t^H for $0 \leq t \leq 1$ is length minimizing among all homotopic paths between the identity and ϕ_1^H.*

Proof: Theorem 2.7 implies that the capacity-area inequality holds for c_{HZ} for all split quasi-cylinders. We can repeat the proof from Proposition 4.4 of [9] to show that it holds for all quasi-cylinders. Thus, c_{HZ} satisfies condition (ii) of Theorem 2.4 for any Hamiltonian H on M if M has dimension two or four. Now, we choose an autonomous H that generates a flow ϕ_t^H which has no non-constant closed trajectories for $0 < t \leq 1$. In order to show that H generates a path which is length minimizing among all homotopic paths, we must show that $c_{HZ}(H) \geq L(H)$ verifying condition (i) of Theorem 2.4. We now invoke Proposition 3.1 from [9]:

Proposition 4.1 *(Lalonde-McDuff) Let M be any symplectic manifold and $H : M \to \mathbf{R}$ be any compactly supported Hamiltonian with no non-constant closed trajectory in time less than 1. Then*

$$c_{HZ}(H) \geq L(H).$$

Proof: We give here a sketch of the proof. Using H, we can construct a specific Hamiltonian $\overline{H}$ on $R_H^-(\frac{\nu}{2})$ and show that $\overline{H} \in \mathcal{H}_{ad}(R_H^-(\frac{\nu}{2}))$. Then, it is easy to show that $m(\overline{H}) \geq m(H) = L(H)$, so $c_{HZ}(R_H^-(\frac{\nu}{2})) \geq L(H)$ and hence $c_{HZ}(H) \geq L(H)$. $\Box$

It follows that ϕ_t^H for $0 \leq t \leq 1$ is length minimizing among all paths homotopic to it with fixed endpoints from the identity to ϕ_1^H, and we are finished with the proof of Theorem 2.8. $\Box$

A Appendix

Here we use Proposition 2.4 and c_G to show two natural paths in $Ham(\mathrm{CP}^2)$ and $Ham(\widetilde{\mathrm{CP}^2})$ given by rotation are length minimizing. The ball embeddings are described explicitly. Of course, Theorem A.1 and Theorem A.5 are also special cases of Theorem 2.8.

A.1 Rotation in CP^2

Theorem A.1 *The path ϕ_t^P for $0 \leq t \leq 1$ in $Ham(\mathrm{CP}^2)$ given by*

$$\phi_t^P[z_0 : z_1 : z_2] = [e^{\pi i t} z_0 : z_1 : z_2]$$

is length minimizing between the identity (ϕ_0^P) and rotation by π radians in the first coordinate (ϕ_1^P).

Proof: To prove this theorem, we will use Gromov capacity c_G and the criteria from Proposition 2.4. To use these criteria, we need the generating Hamiltonian of the path and its length. The Hamiltonian function $P : \mathrm{CP}^2 \to \mathbf{R}$ which generates our path ϕ_t^P is

$$P([z_0 : z_1 : z_2]) = \frac{\pi}{2} \frac{|z_0|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}.$$

Lemma A.2 *The Hamiltonian P defined on CP^2 has $L(P) = \frac{\pi}{2}$.*

Proof: Since P is autonomous,

$$L(P) = \max_{x \in \mathrm{CP}^2} P(x) - \min_{x \in \mathrm{CP}^2} P(x) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

$\square$

Note that the criteria from Proposition 2.4 only tells us if ϕ_t^P will be length minimizing within its homotopy class. However, we use a proposition from [9] to show it is actually globally length minimizing.

Proposition A.3 *(Lalonde-McDuff) Suppose we have a manifold M and a capacity c which satisfies condition (ii) of Proposition 2.4. The path ϕ_t^H is length minimizing amongst all paths with the same endpoints if $c(H) = L(H) \leq \frac{r_1(M)}{2}$.*

The function r_1 is defined as follows: if we let $\mathcal{L} : \pi_1(Ham^c(M)) \to \mathbf{R}$ be defined as

$$\mathcal{L}([\gamma]) = \inf_{\gamma \in [\gamma]} L(\gamma)$$

then

$$r_1(M) = \inf(\{\mathrm{Im}\, \mathcal{L} : \pi_1(Ham^c(M) \to \mathbf{R}\} \cap (0, \infty))$$

if this set is not empty, and ∞ otherwise.

Now, $\pi_1(Ham(\mathrm{CP}^2)) = \mathbf{Z}_3$, generated by rotation through 2π radians in one coordinate [14], specifically the loop

$$\psi_t[z_0 : z_1 : z_2] = [e^{2\pi i t} z_0 : z_1 : z_2]$$

for $0 \leq t \leq 1$. By Theorem 2.8, the path ψ_t for $0 \leq t \leq 1-\epsilon$ is length minimizing among homotopic paths for any $\epsilon > 0$. By a limiting argument, it is easy to see that ψ_t for $0 \leq t \leq 1$ is also length minimizing among homotopic paths. Therefore,

$$r_1(\mathrm{CP}^2) = L(\psi_{t \in [0.1]}) = \pi.$$

By Lemma A.2, $L(\phi_t^P) = L(P) = \frac{\pi}{2} = \frac{r_1(M)}{2}$. Hence, Proposition A.3 implies that if the hypotheses from Proposition 2.4 are satisfied, ϕ_t^P will actually be length minimizing among non-homotopic paths as well as homotopic ones.

Since $L(P) = \frac{\pi}{2}$, we need to show $c_G(P) = \frac{\pi}{2}$. Recall that the capacity of P is the minimum of the capacities of R_P^- and R_P^+, the regions below and above the graph

$$\Gamma_P = \{x, s, t \,|\, P(x) = s\}$$

of P in the six dimensional compact manifold

$$\mathrm{CP}^2 \times [0, \frac{\pi}{2}] \times [0, 1]$$

endowed with the product symplectic form $\tau_0 \oplus ds \wedge dt$.

Since R_P^- and R_P^+ are quasi cylinders with area $L(P) = \frac{\pi}{2}$ and the capacity area inequality holds on CP^2 for c_G, we know that

$$c_G(P) \leq L(P) = \frac{\pi}{2}.$$

To show that $c_G(P) \geq \frac{\pi}{2}$, we show both $c_G(R_P^-) \geq \frac{\pi}{2}$ and $c_G(R_P^+) \geq \frac{\pi}{2}$ by symplectically embedding a 6-ball of radius $1/\sqrt{2} - \epsilon$ into each region.

We explicitly construct a symplectic embedding of $B^6(\frac{1}{\sqrt{2}} - \epsilon)$ into R_P^- and R_P^+. First, we consider

$$R_P^- = \left\{ [z_0 : z_1 : z_2], s, t \,\middle|\, 0 \leq t \leq 1, \ell(t) \leq s \leq \frac{\pi}{2} \frac{|z_0|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \right\}$$

$$\subset \mathrm{CP}^2 \times [0, \frac{\pi}{2}] \times [0, 1]$$

where $\ell(t)$ is some negative function close to 0. In fact, we will embed $B^6(\frac{1}{\sqrt{2}} - \epsilon)$ in the subset of R_P^- where $s \geq 0$. The embedding will be done in two steps: first, we embed a 4-ball in CP^2 and then embed a 2-ball in the two extra graph dimensions.

To understand what is happening geometrically, we identify CP^2 with its image under the moment map of the T^2 action

$$(\theta_0, \theta_1)([z_0 : z_1 : z_2]) = [e^{\pi i \theta_0} z_0 : e^{\pi i \theta_1} z_1 : z_2]$$

with $0 \leq \theta_0, \theta_1 \leq 1$. The moment map for this action $\rho : \mathrm{CP}^2 \to \mathbf{R^2}$ is given by

$$\rho([z_0 : z_1 : z_2]) = \left(\frac{\pi}{2} \frac{|z_0|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}, \frac{\pi}{2} \frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \right)$$

and the image of CP^2 under ρ is the right triangle pictured in Figure 2. The Hamiltonian P is projection onto the horizontal axis and its image is the interval $[0, \frac{\pi}{2}]$.

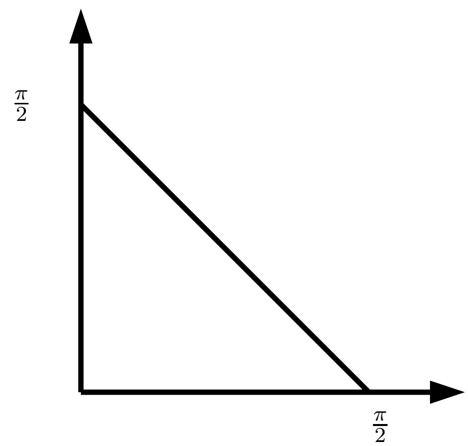

Figure 2: Image of CP^2 under ρ

Let $i^- : \mathbf{C^2} \to \mathrm{CP}^2$ be the map

$$i^-(z_1, z_2) = [\sqrt{1 - |z_1|^2 - |z_2|^2} : z_1 : z_2].$$

Note that i^- restricted to $B^4(s) = \{(z_1, z_2) \,|\, |z_1|^2 + |z_2|^2 \leq s^2\}$ is a symplectic embedding for $s < 1$. The image of i^- composed with ρ is the shaded triangle in Figure 3.

Choose an $r \leq 1/\sqrt{2}$. For any $\epsilon > 0$, we can symplectically embed $B^2(r - \epsilon)$ (the closed 2-ball of radius $r - \epsilon$) into the smaller rectangle in Figure 4 because the area of the ball is $\pi(r - \epsilon)^2$ and the area of the rectangle is $(\frac{\pi}{\sqrt{2}} r)(\frac{2}{\sqrt{2}} r) = \pi r^2$. Denote this mapping by ψ_r^-. Let $R = \frac{1}{\sqrt{2}} - \epsilon$. It is possible to choose the family of maps ψ_r^- so that they fit together to form a smooth map ψ_R^- on $B^2(R)$ such that for $r < R$,

$$\psi_R^-|_{B^2(r)} = \psi_r^-.$$

In particular, this means the images of nested circles under ψ_r^- are disjoint and nested inside the larger rectangle in Figure 4.

We define the map $\Psi^- : B^6(\frac{1}{\sqrt{2}} - \epsilon) \to R_P^-$ by

$$\Psi^-(z_1, z_2, u, v) = (i^-(z_1, z_2), \psi_R^-(u, v))$$

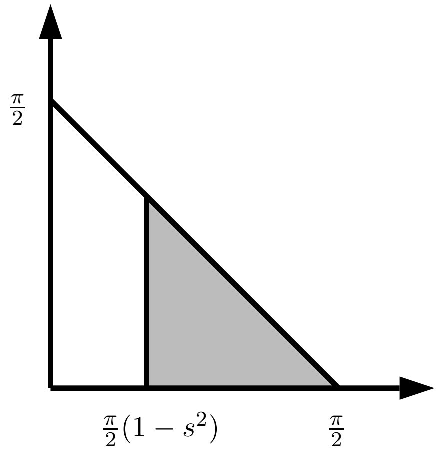

Figure 3: Image of $B^4(s)$ under $\rho \circ i^-$

where the domain coordinates lie in $\mathbf{C} \times \mathbf{C} \times \mathbf{R} \times \mathbf{R}$ and satisfy $|z_1|^2 + |z_2|^2 + u^2 + v^2 \leq (1/\sqrt{2} - \epsilon)^2$. Ψ^- will be the required embedding. We must show that Ψ^- is well defined, i.e. the image of Ψ^- does actually lie in R_P^-. Once this has been demonstrated, it is easy to see that Ψ^- is symplectic since it is the product of two symplectic maps into a symplectic manifold given the product symplectic structure.

Since the map i^- obviously is a well defined embedding, we must only check that for a given point $(z_1, z_2, u, v) \in B^6(\frac{1}{\sqrt{2}} - \epsilon)$, the image of $\psi_R^-(u,v)$ is contained in $[0, \frac{\pi}{2}(1 - |z_1|^2 - |z_2|^2)] \times [0,1] \subset \mathbf{R^2}$. We let $u^2 + v^2 = r^2$ and use the fact that

$$\psi_R^-\big|_{B^2(r)} = \psi_r^-.$$

The height of the rectangle (the second coordinate of the image of ψ_r^-) covers the region

$$\left[\frac{1}{2} - \frac{1}{\sqrt{2}}r, \frac{1}{2} + \frac{1}{\sqrt{2}}r\right]$$

which is contained in the required interval $[0,1]$ for all $r \in [0, \frac{1}{\sqrt{2}}]$. For any given r, the width of the rectangle (the first coordinate of the image of ψ_r^-) covers the region

$$\left[\frac{\pi}{4} + \frac{\pi}{2}r^2 - \frac{\pi}{\sqrt{2}}r, \frac{\pi}{4} + \frac{\pi}{2}r^2\right].$$

As is required, the function $\frac{\pi}{4} + \frac{\pi}{2}r^2 - \frac{\pi}{\sqrt{2}}r$ is greater than zero and decreasing for all values of $r \in [0, \frac{1}{\sqrt{2}}]$. For the final check, we must examine the upper endpoint of the first coordinate of the image of ψ_r^-, $\frac{\pi}{4} + \frac{\pi}{2}r^2$, to ascertain that it is less than or equal to

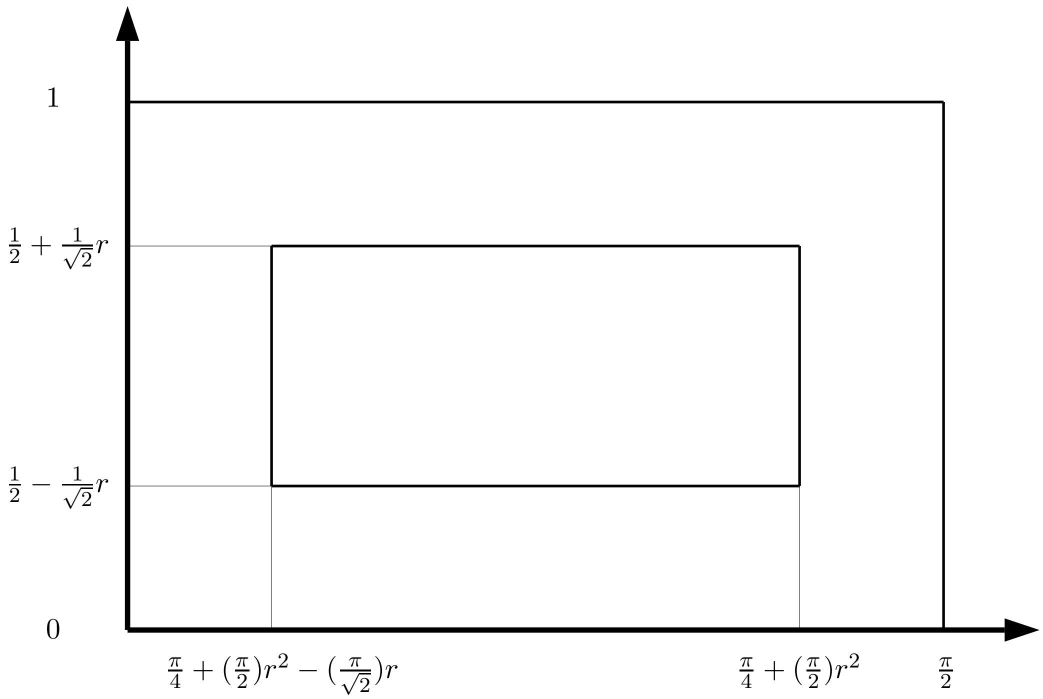

Figure 4: Image of $B^2(r)$ under ψ_r^-

$\frac{\pi}{2}(1-|z_1|^2-|z_2|^2) = P\circ i^-(z_1,z_2)$ for u, v such that $(z_1,z_2,u,v)\in B^6(\frac{1}{\sqrt{2}}-\epsilon)$. This is a simple calculation hinged on the fact that P applied to the image under i^- of any 3-sphere is constant:

$$\begin{aligned}\frac{\pi}{4}+\frac{\pi}{2}r^2 &= \frac{\pi}{4}+\frac{\pi}{2}(u^2+v^2)\\ &\leq \frac{\pi}{4}+\frac{\pi}{2}(\frac{1}{2}-|z_1|^2-|z_2|^2)\\ &= \frac{\pi}{2}(1-|z_1|^2-|z_2|^2).\end{aligned}$$

Hence, the map Ψ^- is a well defined symplectic embedding of $B^6(\frac{1}{\sqrt{2}}-\epsilon)$ into R_P^-.

In a similar manner we can define an embedding Ψ^+ of $B^6(\frac{1}{\sqrt{2}}-\epsilon)$ into R_P^+, the region above Γ_P. We do this in two parts, as before, but now we want to center our ball in the CP^2 portion away from $[1:0:0]$.

Let $i^+ : \mathbf{C^2}\to \mathrm{CP}^2$ be the map

$$i^+(z_1,z_2) = [z_1 : \sqrt{1-|z_1|^2-|z_2|^2} : z_2].$$

Note that i^+ restricted to $B^4(s) = \{z_1,z_2 \,\big|\, |z_1|^2+|z_2|^2\leq s^2\}$ is a symplectic embedding for $s<1$. The image of i^+ composed with ρ is the shaded triangle in Figure 5.

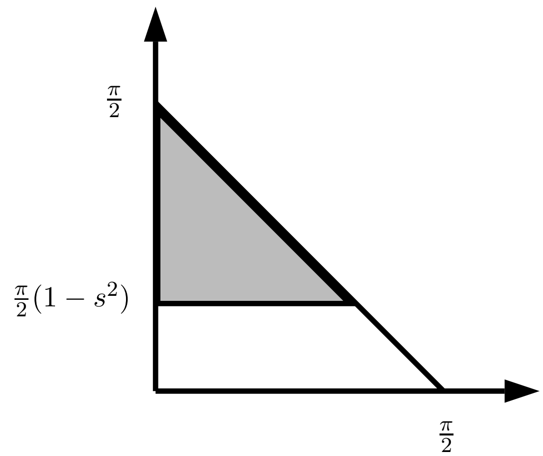

Figure 5: Image of $B^4(s)$ under $\rho \circ i^+$

The map i^+ will be the first part of Ψ^+.

Next, note that we can symplectically embed $B^2(r-\epsilon)$ into the smaller rectangle in Figure 6 because the area of the ball is $\pi(r-\epsilon)^2$ and the area of this rectangle is πr^2.

We denote this mapping by ψ_r^+. As in the previous set up, we may assume that for $r < R$, $\psi_R^+\big|_{B^2(r)} = \psi_r^+$. Then, we define $\Psi^+ : B^6(\frac{1}{\sqrt{2}} - \epsilon) \to R_P^+$ by

$$\Psi^+(z_1, z_2, u, v) = (i^+(z_1, z_2), \psi_R^+(u, v))$$

where the domain coordinates lie in $\mathbf{C} \times \mathbf{C} \times \mathbf{R} \times \mathbf{R}$ and satisfy $|z_1|^2 + |z_2|^2 + u^2 + v^2 \leq (1/\sqrt{2} - \epsilon)^2$. Just as we checked that Ψ is a well defined symplectic embedding, we may verify that Ψ^+ is also a well defined symplectic embedding. $\square$

A.2 Rotation in $\widetilde{\mathrm{CP}^2}$

The next natural path to examine is rotation on the symplectic blow-up $\widetilde{\mathrm{CP}^2}$ of CP^2. For precise details of the definition of $\widetilde{\mathrm{CP}^2}$, see Chapter Six of [12]. Geometrically, $\widetilde{\mathrm{CP}^2}$ can be thought of as the manifold obtained by removing from CP^2 an open 4-ball of radius λ centered at $[1:0:0]$ and collapsing its boundary S^3 along the fibers of the Hopf map. The collapsed S^3, now an S^2, is the exceptional divisor Σ:

$$\Sigma = \left\{ [z_0 : z_1 : z_2] \,\middle|\, \frac{|z_1|^2 + |z_2|^2}{|z_0|^2} = \lambda^2 \right\} / \sim$$

where

$$[z_0 : z_1 : z_2] \sim [w_0 : w_1 : w_2] \text{ if } z_1 w_2 = z_2 w_1.$$

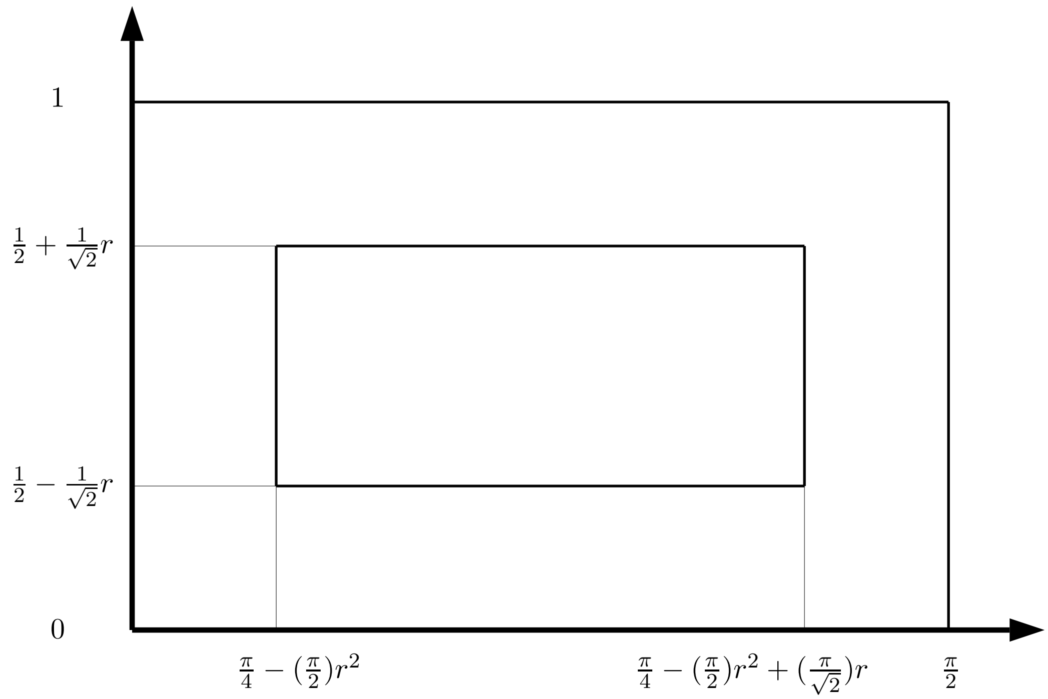

Figure 6: Image of $B^2(r)$ under ψ_r^+

This is the interpretation of $(\widetilde{\mathrm{CP}^2}, \tau_\lambda)$ most often referred to in this paper.

Alternatively, if we think of CP^2 as a 4-ball of radius 1 with the boundary S^3 collapsed along the fibers of the Hopf map, then $(\widetilde{\mathrm{CP}^2}, \tau_\lambda)$ is an annulus $\{(w_0, w_1) \big| (1-\lambda^2) \leq |w_0|^2 + |w_1|^2 \leq 1\}$ with both boundaries collapsed along the Hopf fibers.

The rotation ϕ_t^P on CP^2 in the first homogeneous coordinate descends to a well defined rotation on $\widetilde{\mathrm{CP}^2}$. To check this, it is only necessary to verify that the rotation keeps the set of removed points invariant and that the rotation is well defined under the equivalence imposed on the boundary. In the same way, one can see that the projection of rotation in the second homogeneous coordinate in CP^2

$$\phi_t^Q[z_0 : z_1 : z_2] = [z_0 : e^{\pi i t} z_1 : z_2]$$

is well defined on $\widetilde{\mathrm{CP}^2}$. It is important to realize, however, that the rotation in the first homogeneous coordinate is qualitatively different from rotation in the second: ϕ_t^P fixes each point on Σ, whereas ϕ_t^Q keeps Σ invariant but the points on Σ rotate.

Note that the function P is well defined on $\widetilde{\mathrm{CP}^2}$. When blowing up, we collapse the

boundary of the ball of radius λ along orbits of an S^1 action, and P (defined on CP^2) is invariant under this action. Therefore, P, defined appropriately, is the Hamiltonian function which generates rotation in the first homogeneous coordinate on CP^2 and $\widetilde{\mathrm{CP}^2}$. Hence, we will use P to denote this Hamiltonian function and ϕ_t^P to denote its flow on both manifolds, and it will be clear from context which domain we are considering. Similarly, if we let $Q : \mathrm{CP}^2 \to \mathbf{R}$ be defined as

$$Q[z_0 : z_1 : z_2] = \frac{\pi}{2} \frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}$$

it is clear that Q induces the rotation in the second coordinate ϕ_t^Q on CP^2 and $\widetilde{\mathrm{CP}^2}$.

A.2.1 Rotation in $\widetilde{\mathrm{CP}^2}$ induced by P

Here we begin our treatment of the rotation induced by P applied to $\widetilde{\mathrm{CP}^2}$. In moving from CP^2 to $\widetilde{\mathrm{CP}^2}$, we have altered the domain of P in a consequential way.

Lemma A.4 *The Hamiltonian P defined on $\widetilde{\mathrm{CP}^2}$ has $L(P) = \frac{\pi}{2}(1 - \lambda^2)$.*

Proof: Written out in homogeneous coordinates,

$$\widetilde{\mathrm{CP}^2} = \{[\sqrt{1 - |z_1|^2 - |z_2|^2} : z_1 : z_2] \,\big|\, \lambda^2 \leq |z_1|^2 + |z_2|^2 \leq 1\}$$

with the appropriate equivalence relation on the exceptional divisor. Hence, it is easy to see that

$$L(P) = \max_{x \in \widetilde{\mathrm{CP}^2}} P(x) - \min_{x \in \widetilde{\mathrm{CP}^2}} P(x) = \frac{\pi}{2}(1 - \lambda^2) - 0 = \frac{\pi}{2}(1 - \lambda^2).$$

□

The image of $\widetilde{\mathrm{CP}^2}$ under the map ρ is the quadrilateral in Figure 7. Since the map P is projection onto the horizontal axis, with this quadrilateral as its domain, P has image $[0, \frac{\pi}{2}(1 - \lambda^2)]$. This verifies Lemma A.4.

Theorem A.5 *The path ϕ_t^P for $0 \leq t \leq 1$ in $Ham(\widetilde{\mathrm{CP}^2})$ given by*

$$\phi_t^P[z_0 : z_1 : z_2] = [e^{\pi i t} z_0 : z_1 : z_2]$$

is length minimizing between the identity (ϕ_0^P) and rotation by π radians in the first coordinate (ϕ_1^P).

Proof:

By using the embeddings from the CP^2 case adjusted appropriately, we can show that $c_G(P) = \frac{\pi}{2}(1 - \lambda^2)$. By Proposition 2.4 and Lemma A.4, this will tell us that ϕ_t^P is length minimizing in its homotopy class. Proposition A.3 can be applied in the same way as in the proof of Theorem A.1 to show that ϕ_t^P is actually globally length

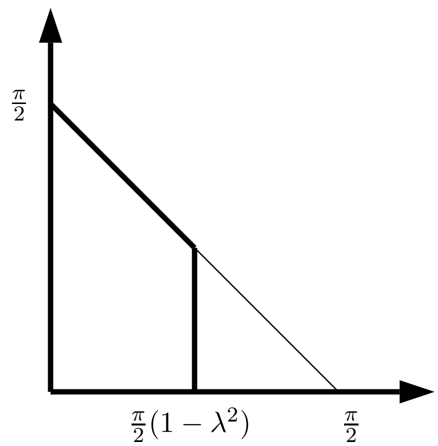

Figure 7: Image of $\widetilde{\mathrm{CP}^2}$ under ρ

minimizing. We omit the details but note that $\pi_1(Ham(\widetilde{\mathrm{CP}^2})) = \mathbf{Z}$ generated by the loop ψ_t described in the first section of this paper, see [1] and [15].

To show that $c_G(R_P^+) \geq \frac{\pi}{2}(1-\lambda^2)$ requires no additional work; we may use the embedding Ψ^+ from the CP^2 case. However, to prove $c_G(R_P^-) \geq \frac{\pi}{2}(1-\lambda^2)$ takes some manipulation. We must produce a new embedding $\Upsilon^- : B^6(\sqrt{\frac{1-\lambda^2}{2}} - \epsilon) \to R_P^-$ because the old embedding, Ψ^-, has in its image some points that were removed under the blow-up.

Consider the open shaded triangle in Figure 8 for some s where $s^2 \in [0, 1-\lambda^2]$.

By Delzant's theorem, the preimage of this set under the map ρ is a symplectic submanifold. This preimage is equal to the set $U_s \subset \widetilde{\mathrm{CP}^2}$ where

$$U_s = \left\{[z_0 : z_1 : z_2] \,\middle|\, |z_0|^2 = (1-\lambda^2-\tau^2), |z_1|^2 < \tau^2, 0 \leq \tau^2 < s^2\right\}.$$

We will prove that there exists a symplectic embedding j_s^- of $B^4(s-\epsilon)$ into U_s. U_s is symplectomorphic to the set $V_s \subset \mathbf{R^4}$ where

$$V_s = \left\{(z_0, B^2(\sqrt{1-\lambda^2-|z_0|^2})) \,\middle|\, z_0 \in \mathbf{C}, 1-\lambda^2-s^2 < |z_0|^2 < 1-\lambda^2\right\}.$$

V_s is just a set of 2-balls fibered over an annulus. If we cut this annulus to make it a rectangle (this does not change the symplectic capacity), we arrive at the set

$$T_s = \left\{(x, y, B^2(\sqrt{s^2-y^2})) \,\middle|\, 0 \leq x < \pi s, 0 \leq y < s\right\} \subset \mathbf{R^4}.$$

T_s is a generalized trapezoid, that is it consists of balls fibered over a rectangle. It is not hard to show that the capacity of T_s is the same as the capacity of the more standard

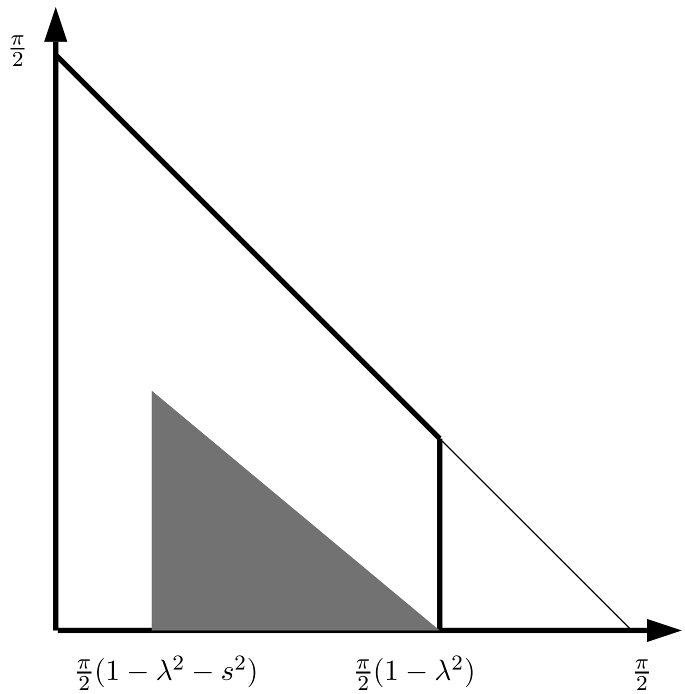

Figure 8: Image of $B^4(s)$ under $\rho \circ j^-$

trapezoid

$$T^4(\pi s^2) = \left\{ (x, y, B^2(\sqrt{s^2 - \frac{y}{\pi}}) \,\middle|\, 0 \leq x < 1, 0 \leq y < \pi s^2 \right\}.$$

In Lemma 3.6 of [9], it is shown that the capacity of $T^4(\pi s^2)$ is equal to the capacity of $B^4(s)$. Hence,

$$c_G(U_s) = c_G(T_s) = c_G(T^4(\pi s^2)) = c_G(B^4(s)) = \pi s^2$$

and we can embed $B^4(s - \epsilon)$ into U_s for any $\epsilon > 0$. Call this embedding j_s^-. Consider the family of maps $j_s^- : B^4(s - \epsilon) \to U_s$ for all $0 \leq s \leq \sqrt{1 - \lambda^2}$. Let

$$S = \sqrt{\frac{1 - \lambda^2}{2}} - \epsilon.$$

Without loss of generality, we may assume that the family of maps j_s satisfies

$$j_S^-|_{B^4(s)} = j_s^-$$

for $s \leq S$, so that 3-spheres of constant radius appear as vertical lines in the moment map picture. To be precise, if $(w_o, w_1) \in \mathbf{C^2}$ and $|w_0|^2 + |w_1|^2 = s^2$, then $\rho \circ j_S^-(w_0, w_1)$ lies on the vertical line through the point $(\frac{\pi}{2}(1 - \lambda^2 - s^2), 0)$. Thus, P applied to the image of 3-spheres under j_S^- is constant.

Now, we have an embedding j_S^- from $B^4(\sqrt{\frac{1-\lambda^2}{2}} - \epsilon)$ into $\widetilde{\mathrm{CP}^2}$. Our next task is to work with the other two dimensions and construct Υ^-.

Fix an $r < \sqrt{\frac{1-\lambda^2}{2}}$. We can symplectically embed $B^2(r)$ into the smaller rectangle in Figure 9 because the area of the ball is πr^2 and the area of the rectangle is

$$\left(\frac{\pi}{2}(\sqrt{1-\lambda^2})r\right)\left(\frac{2}{\sqrt{2(1-\lambda^2)}}\right) = \pi r^2.$$

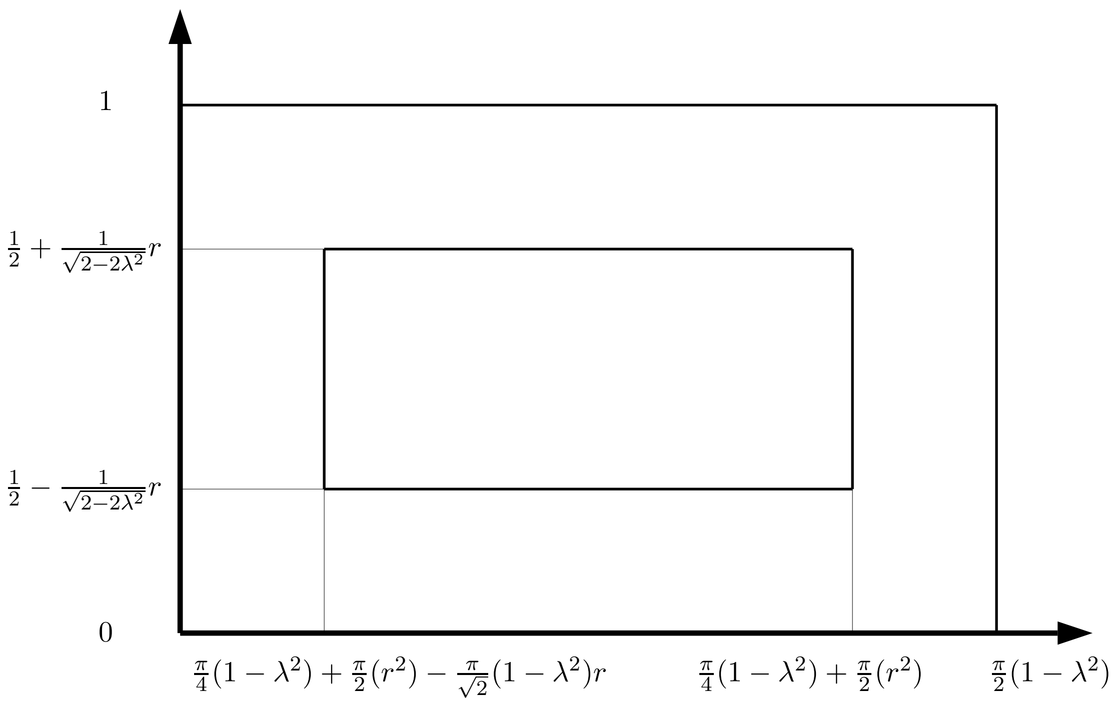

Figure 9: Image of $B^2(r)$ under υ_r^-

Denote this embedding by υ_r^-. As before, we assume that for $r < S$, $\upsilon_S^-\big|_{B^2(r)} = \upsilon_r^-$, and define $\Upsilon^- : B^6(S) \to R_P^-$ by

$$\Upsilon^-(w_0, w_1, u, v) = \left(j^-(w_0, w_1), \upsilon_S^-(u, v)\right).$$

Using the fact that P is constant along the image under j^- of 3-spheres, it is routine to check that in fact Υ^- is well defined. $\square$

A.2.2 Rotation in $\widetilde{\mathrm{CP}^2}$ induced by Q

Now, recall the Hamiltonian function $Q : \mathrm{CP}^2 \to \mathbf{R}$ given by

$$Q([z_0 : z_1 : z_2]) = \frac{\pi}{2} \frac{|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}.$$

It is easy to check that $L(Q) = \frac{\pi}{2}$ and that the flow of Q is the path in $Ham(\mathrm{CP}^2)$

$$\phi_t^Q [z_0 : z_1 : z_2] = [z_0 : e^{\pi i t} z_1 : z_2].$$

Q descends to a well defined function on $\widetilde{\mathrm{CP}^2}$, and its flow descends to a well defined rotation on $\widetilde{\mathrm{CP}^2}$.

Lemma A.6 *The Hamiltonian Q defined on $\widetilde{\mathrm{CP}^2}$ has $L(Q) = \frac{\pi}{2}$.*

Proof: Written out in homogeneous coordinates,

$$\widetilde{\mathrm{CP}^2} = \{[\sqrt{1 - |z_1|^2 - |z_2|^2} : z_1 : z_2] \,\big|\, \lambda^2 \leq |z_1|^2 + |z_2|^2 \leq 1\}$$

with the appropriate equivalence relation on the exceptional divisor. Hence, it is easy to see that

$$L(Q) = \max_{x \in \widetilde{\mathrm{CP}^2}} Q(x) - \min_{x \in \widetilde{\mathrm{CP}^2}} Q(x) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.\square$$

Recall that the image of ρ applied to $\widetilde{\mathrm{CP}^2}$ is the quadrilateral depicted in Figure 7. The map Q defined on $\widetilde{\mathrm{CP}^2}$ is projection onto the vertical axis in this picture and has image $[0, \frac{\pi}{2}]$, verifying Lemma A.6.

We could show that the path ϕ_t^Q for $0 \leq t \leq 1$ defined on CP^2 is length minimizing by using the argument from Theorem A.1. However, we cannot use the arguments from Theorem A.5 to show that ϕ_t^Q is length minimizing on $\widetilde{\mathrm{CP}^2}$. Lemma A.6 tells us that the length of Q does not decrease when going from CP^2 to $\widetilde{\mathrm{CP}^2}$. However, the volume of the manifold $\widetilde{\mathrm{CP}^2}$ is less than the volume of CP^2. There is not a straight forward way to embed large enough 6-balls to show that $c_G(Q) = \frac{\pi}{2}$ on $\widetilde{\mathrm{CP}^2}$, i.e. 6-balls of raduis $\frac{1}{\sqrt{2}}$. (Recall that in the proof of Theorem A.5 for ϕ_t^P we only had to embed balls of radius close to $\sqrt{\frac{1-\lambda^2}{2}}$.)

These two rotations, ϕ_t^P and ϕ_t^Q, are essentially the only two different types of rotations of CP^2 which descend to rotations on $\widetilde{\mathrm{CP}^2}$. In order for any rotation to descend properly from CP^2 to $\widetilde{\mathrm{CP}^2}$, $[1 : 0 : 0]$ must be a fixed point of the rotation in CP^2. A rotation in CP^2 has one isolated fixed point and a fixed sphere, e.g. ϕ_t^P has an isolated fixed point at $[1 : 0 : 0]$ and a fixed sphere consisting of the points of the form

$[0 : z_1 : z_2] \subset \mathrm{CP}^2$. Since $[1 : 0 : 0]$ is the isolated fixed point of ϕ_t^P and a point on the fixed sphere of ϕ_t^Q, we have accounted for both types of rotations.

Because $L(Q)$ does not decrease when moving from CP^2 to $(\widetilde{\mathrm{CP}^2}, \tau_\lambda)$, we cannot use the Gromov capacity to show that the rotation induced by Q on $\widetilde{\mathrm{CP}^2}$ is length minimizing.

References

[1] M. Abreu and D. McDuff, Topology of symplectomorphism groups of rational ruled surfaces, preprint (1997).

[2] M. Bialy and L. Polterovich, Geodesics of Hofer's metric on the group of Hamiltonian diffeomorphisms, *Duke J. Math.* **76** (1994), 273-292.

[3] H. Hofer, Estimates for the energy of a symplectic map. *Comm. Math. Helv.*, **68** (1993), 48-72.

[4] H. Hofer, On the topological properties of symplectic maps. *Proceedings of the Royal Society of Edinburgh*, **115** (1990), A:25-38.

[5] H. Hofer and C. Viterbo, The Weinstein Conjecture in the Presence of Holomorphic Spheres, *Comm. on Pure and Applied Math.* **XLV** (1992), 583-622.

[6] H. Hofer and E. Zehnder, *Symplectic Invariants and Hamiltonian Dynamics*, Birkhauser, Boston, MA (1994).

[7] Y.Karshon, Periodic Hamiltonian Flows in Four Dimensional Manifolds, preprint (1998).

[8] F. Lalonde and D. McDuff, Geometry of Symplectic Energy, *Annals of Mathematics* **141:2** (1996), 349-371.

[9] F. Lalonde and D. McDuff, Hofer's L^∞-geometry: energy and stability of Hamiltonian flows parts I and II, *Invent. Math.* **122** (1995), 1-33 and 35-69.

[10] G. Liu and G. Tian, Weinstein Conjecture and GW Invariants, preprint (1997).

[11] D. McDuff, The Virtual Moduli Cycle, preprint (1998).

[12] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*, Oxford University Press, Oxford, England (1995).

[13] D. McDuff and D. Salamon, *J-Holomorphic Curves and Quantum Cohomology*, University Lecture Series **6**, American Mathematical Society (1994).

[14] L. Polterovich, Gromov's K-area and symplectic rigidity, *Geom. and Funct. Analysis* **6** (1996), 726-739.

[15] L. Polterovich, Hamiltonian loops and Arnold's Principle, *Am. Math. Soc. Transl.* (2) **180** (1997), 181-187.

[16] L. Polterovich, Hofer's diameter and Lagrangian intersections, *IMRN* **4** (1998), 217-223.

[17] K. Siburg, New minimal geodesics in the group of symplectic diffeomorphisms, *Calc. Var* **3** (1995), 299-309.

[18] I. Ustilovsky, Conjugate points on geodesics of Hofer's metric. *Diff. Geometry and its Appl.* **6** (1994), 327-342.

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